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# A note on a canonical dynamical $r$-matrix

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**Abstract**

It is well known that a classical dynamical $r$-matrix can be associated with every finite-dimensional self-dual Lie algebra $\mathcal{G}$ by the definition $R(\omega) := f(\text{ad }\omega)$, where $\omega \in \mathcal{G}$ and $f$ is the holomorphic function given by $f(z) = \frac{1}{2}\coth(z/2) - 1/z$ for $z \in \mathbb{C} \setminus 2\pi \mathrm{i}\mathbb{Z}^*$. We present a new, direct proof of the statement that this canonical $r$-matrix satisfies the modified classical dynamical Yang–Baxter equation on $\mathcal{G}$.

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## 1. Introduction

Dynamical generalizations of the Yang–Baxter equations and the associated algebraic structures are a focus of current interest due to their applications in the theory of integrable systems and other areas of mathematical physics and pure mathematics (see [1] for a review). The present paper contains a detailed study of a particular dynamical $r$-matrix, which is an important special case of the classical dynamical $r$-matrices introduced in [2].

Let us recall that dynamical $r$-matrices in the sense of Etingof and Varchenko [2] are associated with any subalgebra $\mathcal{H}$ of any (complex or real) Lie algebra $\mathcal{G}$. By definition, a dynamical $r$-matrix is a (holomorphic or smooth) $\mathcal{G} \otimes \mathcal{G}$-valued function on an open subset $\check{\mathcal{H}}^*$ of the dual space $\mathcal{H}^*$ of $\mathcal{H}$ subject to the following three conditions. First, $r$ must satisfy the modified classical dynamical Yang–Baxter equation (mCDYBE):

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] + T_j^1 \frac{\partial r_{23}}{\partial \omega_j} - T_j^2 \frac{\partial r_{13}}{\partial \omega_j} + T_j^3 \frac{\partial r_{12}}{\partial \omega_j} = \varphi \tag{1.1}$$

where $\varphi$ is some constant, $\mathcal{G}$-invariant element of $\mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$. The $\omega_j$ are coordinates on $\mathcal{H}^*$ with respect to a basis $\{T_j\}$ of $\mathcal{H}$, and the usual tensorial notation as well as the summation convention are used. The second condition is that $(r + r^{\mathrm{T}})$, where $(X_a \otimes Y^a)^{\mathrm{T}} = Y^a \otimes X_a$, is a $\mathcal{G}$-invariant constant. The third condition requires the map $r : \check{\mathcal{H}}^* \to \mathcal{G} \otimes \mathcal{G}$ to be equivariant

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with respect to the (coadjoint and adjoint) infinitesimal actions of $\mathcal{H}$ on the corresponding spaces. The mCDYBE becomes the CDYBE for $\varphi = 0$.

In most applications $\mathcal{G}$ is a simple Lie algebra and $\mathcal{H}$ is (a subalgebra of) a Cartan subalgebra. Another interesting special case is obtained by taking $\mathcal{H} := \mathcal{G}$. We consider this latter case, and allow $\mathcal{G}$ to be any self-dual Lie algebra for which $\mathcal{G}^*$ can be identified with $\mathcal{G}$ by means of an invariant scalar product $\langle\ ,\ \rangle$. We here study the dynamical $r$-matrix given by the formula

$$r : \omega \mapsto r(\omega) := \langle T_j, f(\mathrm{ad}\ \omega)T_k\rangle T^j \otimes T^k \qquad \omega \in \check{\mathcal{G}} \tag{1.2}$$

where $\check{\mathcal{G}} \subset \mathcal{G}$ is an open subset, $\{T_j\}$ and $\{T^k\}$ denote dual bases of $\mathcal{G}$, $\langle T_j, T^k\rangle = \delta_j^k$, and $f$ is the complex analytic function defined by

$$f(z) := \frac{1}{2}\coth\frac{z}{2} - \frac{1}{z} \qquad z \in \mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}^*. \tag{1.3}$$

It is known that this $r$-matrix is a solution of the mCDYBE (1.1) for $\mathcal{H} = \mathcal{G} \simeq \mathcal{G}^*$ with

$$\varphi = -\tfrac{1}{4} f_{jk}^l T^j \otimes T^k \otimes T_l \qquad [T_j, T_k] = f_{jk}^l T_l. \tag{1.4}$$

If $\mathcal{G}$ is a simple Lie algebra, then the mCDYBE for $r$ in (1.2) follows from a general result (theorem 3.14) in [2]. Remarkably, this $r$-matrix came to light naturally in two different applications, namely in the context of equivariant cohomology [3] and in the description of a Poisson structure on the chiral WZNW phase space compatible with classical $\mathcal{G}$-symmetry [4]. A further reason for which the $r$-matrix in (1.2) is important is that it can be reduced to certain self-dual subalgebras of $\mathcal{G}$, and thereby serves as a common 'source' for a large family of dynamical $r$-matrices [5]. We call it the canonical $r$-matrix of the self-dual Lie algebra $\mathcal{G}$.

The authors of [3] assumed $\mathcal{G}$ to be compact, while in [4] $\mathcal{G}$ was taken to be a simple Lie algebra. In these papers the mCDYBE for the canonical $r$-matrix was proved, independently, using the additional assumption that $\omega$ is near to zero, so that $f(\mathrm{ad}\ \omega)$ is given with the aid of the power series expansion of $f(z)$ around $z = 0$. Though this is not obvious, the proofs found in [3, 4] (see also [6, 7]) can in fact be adapted to cover the case of a general self-dual Lie algebra as well. In this case, a different proof of the mCDYBE appeared in [8]. This proof is indirect and uses the restriction of $\omega$ to a neighbourhood of the origin. The maximal domain of definition of $f(\mathrm{ad}\ \omega)$ contains all $\omega$ for which the eigenvalues of ad $\omega$ lie in $\mathbb{C} \setminus 2\pi\mathrm{i}\mathbb{Z}^*$. Although the above-mentioned local proofs and the analyticity of $r(\omega)$ together imply the mCDYBE on this domain, it could be enlightening to have an alternative direct proof, too.

The purpose of this paper is to present a direct proof of the mCDYBE for the canonical $r$-matrix. As opposed to the local power series expansion around 0, we here use the well known [9] holomorphic functional calculus of linear operators to define $f(\mathrm{ad}\ \omega)$, and thus our proof is valid globally on the maximal domain of the 'dynamical variable' $\omega$. An advantage of our proof is that it also yields a uniqueness result for the holomorphic function $f(z)$ that enters the definition of the $r$-matrix in (1.2). Namely, on taking formula (1.2) as an ansatz the mCDYBE translates into a functional equation (equation (C.1)) for the holomorphic function $f$ that has (1.3) as its unique solution under certain further natural conditions. Despite its length, its elementary character and the uniqueness result that it implies might justify presenting our proof.

After this introduction, the paper consists of two sections and three appendices. The proof of the mCDYBE is described in section 2. It relies on some technical material collected in the appendices. Appendix A recalls relevant basics of the functional calculus from [9]—if necessary, the reader might consult that first—while the other two appendices should be consulted as referred to in the proof. Section 3 is devoted to a discussion of consequences of the proof, including the above-mentioned uniqueness result for the function $f$, and some comments.

## 2. Proof of the mCDYBE for the canonical $r$-matrix

Let $\mathcal{G}$ be a finite-dimensional complex Lie algebra equipped with an invariant, symmetric, nondegenerate bilinear form $\langle\ ,\ \rangle$. For the structure of such Lie algebras, see e.g. [10]. We call these Lie algebras self-dual, since we identify $\mathcal{G}$ with $\mathcal{G}^*$ by means of the 'scalar product' $\langle\ ,\ \rangle$. Defining the transpose $A^{\mathrm{T}}$ of an operator $A \in \mathrm{End}(\mathcal{G})$ by $\langle A^{\mathrm{T}}X, Y\rangle = \langle X, AY\rangle$ $(\forall X, Y \in \mathcal{G})$, the invariance property of $\langle\ ,\ \rangle$ means that $(\mathrm{ad}\ \omega)^{\mathrm{T}} = -\mathrm{ad}\ \omega$ $(\forall \omega \in \mathcal{G})$, where $(\mathrm{ad}\ \omega)(X) = [\omega, X]$.

Consider a map $r : \check{\mathcal{G}} \to \mathcal{G} \otimes \mathcal{G}$, where $\check{\mathcal{G}} \subset \mathcal{G}$ is a nonempty open subset. Then there exists a unique map $R : \check{\mathcal{G}} \to \mathrm{End}(\mathcal{G})$ for which

$$r(\omega) = \langle T_j, R(\omega)T_k\rangle T^j \otimes T^k \qquad \forall \omega \in \check{\mathcal{G}} \tag{2.1}$$

where $\{T_j\}$ and $\{T^k\}$ denote dual bases of $\mathcal{G}$. The directional derivatives of $R$ are given by

$$(\nabla_S R)(\omega) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} R(\omega + tS) \qquad \forall S \in \mathcal{G}, \omega \in \check{\mathcal{G}} \tag{2.2}$$

and the 'gradient' of $R$ is defined by

$$\langle X, (\nabla R)(\omega)Y\rangle := T^j \langle X, (\nabla_{T_j} R)(\omega)Y\rangle \qquad \forall X, Y \in \mathcal{G}, \omega \in \check{\mathcal{G}}. \tag{2.3}$$

If $r$ is antisymmetric, i.e., $R^{\mathrm{T}}(\omega) = -R(\omega)$, then the mCDYBE (1.1) for $r$ with $\varphi$ in (1.4) is in fact equivalent to the following equation for $R$:

$$\begin{aligned}&\tfrac{1}{4}[X, Y] + [R(\omega)X, R(\omega)Y] - R(\omega)\,([R(\omega)X, Y] + [X, R(\omega)Y]) + \langle X, (\nabla R)(\omega)Y\rangle \\ &\quad + (\nabla_Y R)(\omega)X - (\nabla_X R)(\omega)Y = 0 \qquad \forall X, Y \in \mathcal{G}, \omega \in \check{\mathcal{G}}.\end{aligned} \tag{2.4}$$

The $\mathcal{G}$-equivariance of the map $r : \check{\mathcal{G}} \to \mathcal{G} \otimes \mathcal{G}$ can be expressed as

$$(\nabla_{[S,\omega]} R)(\omega) = [\mathrm{ad}\ S, R(\omega)] \qquad \forall S \in \mathcal{G}, \omega \in \check{\mathcal{G}}. \tag{2.5}$$

Having made these remarks, we are ready to study the canonical $r$-matrix. From now on we use

$$\check{\mathcal{G}} := \left\{ \omega \in \mathcal{G} \mid \sigma(\mathrm{ad}\ \omega) \cap 2\pi\mathrm{i}\mathbb{Z}^* = \emptyset \right\} \tag{2.6}$$

which is a nonempty open subset in $\mathcal{G}$. Here and below, $\sigma(\mathrm{ad}\ \omega)$ denotes the spectrum of $\mathrm{ad}\ \omega$ $(\forall \omega \in \mathcal{G})$, and sometimes we use the notation $\bar{\omega} := \mathrm{ad}\ \omega$ for brevity. With the aid of the familiar holomorphic functional calculus (see appendix A), we can define an operator-valued dynamical $r$-matrix $R\colon \check{\mathcal{G}} \to \mathrm{End}(\mathcal{G})$ by

$$\omega \mapsto R(\omega) := f(\mathrm{ad}\ \omega) = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}\xi\ f(\xi)(\xi I - \mathrm{ad}\ \omega)^{-1} \tag{2.7}$$

where $f$ is given in (1.3). The curve $C$ encircles each eigenvalue of $\mathrm{ad}\ \omega$ and $I$ is the identity operator on $\mathcal{G}$. Now our main theorem can be formulated as follows.

**Theorem 1.** *The mapping (2.7) with $f$ in (1.3) defines an antisymmetric $r$-matrix which satisfies the equivariance condition (2.5) and the mCDYBE given by (2.4).*

The antisymmetry of the $r$-matrix follows from (2.7) by using that $f$ is an odd function, and the equivariance condition (2.5) is also an immediate consequence of (2.7) (cf. (A.3)). Before verifying (2.4), we gather some useful information and lemmas that make the calculations easier.

Let $\omega$ be an arbitrary fixed element of $\check{\mathcal{G}}$. For every $\lambda \in \mathbb{C}$, let $b_\lambda := \mathrm{ad}\ \omega - \lambda I = \bar{\omega} - \lambda I \in \mathrm{End}(\mathcal{G})$. Thanks to the derivation property of $\mathrm{ad}\ \omega$, the $b_\lambda$ enjoy the identities

$$b^n_{\alpha+\beta}[X, Y] = \sum_{j=0}^{n} \binom{n}{j} \left[b^j_\alpha X, b^{n-j}_\beta Y\right] \qquad \forall X, Y \in \mathcal{G}, \forall \alpha, \beta \in \mathbb{C}. \tag{2.8}$$

By means of the $\mathcal{G} = \oplus_{\lambda\in\sigma(\bar{\omega})} N_\lambda$ Jordan decomposition, where $N_\lambda = \mathrm{Ker}(b_\lambda^{\nu(\lambda)})$ (see appendix A), the $r$-matrix (2.7) can be written as

$$R(\omega) = f(\bar{\omega}) = \sum_{\lambda\in\sigma(\bar{\omega})} \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k E_\lambda. \tag{2.9}$$

We can regard this equation as the application of (A.4) to the operator ad $\omega$. Here $E_\lambda \in \mathrm{End}(\mathcal{G})$ means the projection corresponding to the subspace $N_\lambda$. Note also that $[N_\lambda, N_\mu] \subset N_{\lambda+\mu}$ is implied by (2.8), with $N_\mu = \{0\}$ for any $\mu \notin \sigma(\bar{\omega})$.

The mCDYBE (2.4) is linear in $X$ and $Y$. Therefore it is enough to prove this equation when $X \in N_\lambda$, $Y \in N_\mu$ are arbitrary elements of the subspaces associated with the eigenvalues $\lambda, \mu \in \sigma(\bar{\omega})$. So, from now on let $\lambda, \mu$ be arbitrary, fixed eigenvalues of $\bar{\omega}$ and $X \in N_\lambda$, $Y \in N_\mu$ arbitrary, but fixed vectors. Applying the $r$-matrix (2.9) to these vectors, we obtain

$$\begin{aligned} R(\omega)X &= f(\bar{\omega})X = \sum_{k=0}^{\nu(\lambda)-1} \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k X \\ R(\omega)Y &= f(\bar{\omega})Y = \sum_{l=0}^{\nu(\mu)-1} \frac{f^{(l)}(\mu)}{l!} b_\mu^l Y. \end{aligned} \tag{2.10}$$

In the following four lemmas we calculate the various terms of the mCDYBE (2.4) in a form that will prove convenient for verifying this equation. In all expressions containing $\left[b_\lambda^k X, b_\mu^l Y\right]$ it is understood that the indices $k, l$ vary as in (2.10).

**Lemma 2.** *If* $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, *then*

$$\frac{1}{4}[X, Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{1}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \tag{2.11}$$

$$[f(\bar{\omega})X, f(\bar{\omega})Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha) f(\beta) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \tag{2.12}$$

$$f(\bar{\omega})[f(\bar{\omega})X, Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha+\beta) f(\alpha) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \tag{2.13}$$

$$f(\bar{\omega})[X, f(\bar{\omega})Y] = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha+\beta) f(\beta) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.14}$$

**Proof.** First, identity (C.3) from appendix C leads immediately to (2.11) as

$$\frac{1}{4}[X, Y] = \frac{1}{4}\left[b_\lambda^0 X, b_\mu^0 Y\right] = \sum_{k,l} \frac{\delta_{k,0}\delta_{l,0}}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} = \sum_{k,l} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{1}{4} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.15}$$

Second, with the aid of (2.10) and (C.4), we easily obtain (2.12)

$$\begin{aligned} [f(\bar{\omega})X, f(\bar{\omega})Y] &= \sum_{k,l} f^{(k)}(\lambda) f^{(l)}(\mu) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \\ &= \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} f(\alpha) f(\beta) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \end{aligned} \tag{2.16}$$

Third, the calculation of

$$f(\bar{\omega})[f(\bar{\omega})X, Y] = f(\bar{\omega})\left[\sum_k \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k X, Y\right] \tag{2.17}$$

goes as follows. Since $\left[\sum_k \frac{f^{(k)}(\lambda)}{k!} b_\lambda^k X, Y\right] \in N_{\lambda+\mu}$, equation (2.10) yields

$$\begin{aligned} f(\bar\omega)[f(\bar\omega)X, Y] &= \sum_{k,l} \frac{f^{(k)}(\lambda) f^{(l)}(\lambda+\mu)}{k!l!} b_{\lambda+\mu}^l \left[b_\lambda^k X, Y\right] \\ &= \sum_{k,l} \frac{f^{(k)}(\lambda) f^{(l)}(\lambda+\mu)}{k!l!} \sum_{j=0}^{l} \binom{l}{j} \left[b_\lambda^{k+l-j} X, b_\mu^j Y\right] \end{aligned} \tag{2.18}$$

where we used (2.8). Introducing a new variable $s := k + l$ for the summation, we have

$$\begin{aligned} f(\bar\omega)\left[f(\bar\omega)X, Y\right] &= \sum_s \sum_{j=0}^{s} \sum_{l=j}^{s} \binom{l}{j} \frac{f^{(s-l)}(\lambda) f^{(l)}(\lambda+\mu)}{(s-l)!l!} \left[b_\lambda^{s-j} X, b_\mu^j Y\right] \\ &= \sum_s \sum_{j=0}^{s} \sum_{l=0}^{s-j} \binom{l+j}{j} \frac{f^{(j+l)}(\lambda+\mu) f^{(s-j-l)}(\lambda)}{(l+j)!(s-j-l)!} \left[b_\lambda^{s-j} X, b_\mu^j Y\right] \\ &= \sum_s \sum_{j=0}^{s} \frac{1}{j!(s-j)!} \sum_{l=0}^{s-j} \binom{s-j}{l} f^{(j+l)}(\lambda+\mu) f^{(s-j-l)}(\lambda) \left[b_\lambda^{s-j} X, b_\mu^j Y\right]. \end{aligned} \tag{2.19}$$

Using the Leibniz rule and introducing new summation variables as $l := j$, $k := s - j$, we obtain

$$\begin{aligned} f(\bar\omega)[f(\bar\omega)X, Y] &= \sum_s \sum_{j=0}^{s} \frac{1}{j!(s-j)!} \frac{\mathrm{d}^{s-j}}{\mathrm{d}\xi^{s-j}}\bigg|_{\xi=\lambda} f^{(j)}(\xi+\mu) f(\xi) \left[b_\lambda^{s-j} X, b_\mu^j Y\right] \\ &= \sum_{k,l} \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\bigg|_{\xi=\lambda} f^{(l)}(\xi+\mu) f(\xi) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \end{aligned} \tag{2.20}$$

By (C.5), this gives (2.13). Finally, equation (2.14) is trivial consequence of (2.13). □

**Lemma 3.** *If* $\lambda, \mu \in \sigma(\bar\omega)$, $X \in N_\lambda$, $Y \in N_\mu$, *then*

$$\langle X, (\nabla R)(\omega)Y\rangle = -\sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{f(\alpha)+f(\beta)}{\alpha+\beta} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.21}$$

**Proof.** We obtain directly from the definitions (2.2), (2.3), (2.7) (see also (A.3)) that

$$\langle X, (\nabla R)(\omega) Y\rangle = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}\xi\, f(\xi)\, T^j \langle X, \rho_\xi(\bar\omega)[T_j, \rho_\xi(\bar\omega) Y]\rangle \tag{2.22}$$

where $\rho_\xi(\bar\omega) = (\xi I - \bar\omega)^{-1}$. By using that $\rho_\xi(\bar\omega)^{\mathrm{T}} = -\rho_{-\xi}(\bar\omega)$ and the invariance of $\langle\ ,\ \rangle$, this expression is easily converted into

$$\langle X, (\nabla R)(\omega) Y\rangle = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}\xi\, f(\xi) \left[\rho_{-\xi}(\bar\omega) X, \rho_\xi(\bar\omega) Y\right]. \tag{2.23}$$

We can apply the functional calculus to the holomorphic function $\rho_\xi : (\mathbb{C} \setminus \{\xi\}) \to \mathbb{C}$ defined by $\rho_\xi : z \mapsto (\xi - z)^{-1}$. Thus we have

$$\rho_{-\xi}(\bar\omega) X = \sum_k \frac{\rho_{-\xi}^{(k)}(\lambda)}{k!} b_\lambda^k X \qquad \rho_\xi(\bar\omega) Y = \sum_l \frac{\rho_\xi^{(l)}(\mu)}{l!} b_\mu^l Y \tag{2.24}$$

similarly to (2.10). Since $\rho_{-\xi}^{(k)}(\lambda) = k!(-\xi-\lambda)^{-(k+1)} = (-1)^{k+1} \rho_\xi(-\lambda)$, this leads to

$$\langle X, (\nabla R)(\omega) Y\rangle = \sum_{k,l} \left( \frac{(-1)^{k+1}}{2\pi\mathrm{i}} \int_C \mathrm{d}\xi\, f(\xi)\, \rho_\xi^{(k)}(-\lambda)\, \rho_\xi^{(l)}(\mu) \right) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.25}$$

Now our task is to determine these integrals. Obviously, two different cases can appear. When $-\lambda = \mu$, the integrands have poles only at the point $\mu$. Alternatively, when $-\lambda \neq \mu$, the integrands have poles at the point $-\lambda$ and at the point $\mu$.

*The* $\lambda + \mu = 0$ *case.* In this case $\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) = k!l!\,(\xi-\mu)^{-(k+l+1)-1}$. Thanks to Cauchy's theorem, the integrals can be written as

$$\frac{(-1)^{k+1}}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\, f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) = \frac{(-1)^{k+1}\,k!l!}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\,\frac{f(\xi)}{(\xi-\mu)^{(k+l+1)+1}}$$
$$= \frac{(-1)^{k+1}\,k!l!}{(k+l+1)!}f^{(k+l+1)}(\mu) = -\lim_{(\alpha,\beta)\to(\lambda,\mu)}\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\frac{f(\alpha)+f(\beta)}{\alpha+\beta} \tag{2.26}$$

where we used the identity (C.8). Thus (2.21) is valid in this case.

*The* $\lambda + \mu \neq 0$ *case.* By $C_\alpha$ we denote a sufficiently small circle around the eigenvalue $\alpha \in \sigma(\bar{\omega})$, which encircles this point in the positive sense. Using Cauchy's theorem in (2.25), we can write

$$\frac{(-1)^{k+1}}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\, f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu)$$
$$= (-1)^{k+1}\left\{\frac{1}{2\pi\mathrm{i}}\int_{C_\mu}\mathrm{d}\xi\, f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu)\right.$$
$$\left.+\frac{1}{2\pi\mathrm{i}}\int_{C_{-\lambda}}\mathrm{d}\xi\, f(\xi)\,\rho_\xi^{(l)}(\mu)\,\rho_\xi^{(k)}(-\lambda)\right\}$$
$$= (-1)^{k+1}\left\{\left.\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\right|_{\xi=\mu} f(\xi)\,(-1)^{k+1}\,\rho_{-\lambda}^{(k)}(\xi) + \left.\frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\right|_{\xi=-\lambda} f(\xi)\,(-1)^{l+1}\,\rho_\mu^{(l)}(\xi)\right\}$$
$$= (-1)^k\left\{\sum_{a=0}^{l}(-1)^k\binom{l}{a}f^{(a)}(\mu)\,\rho_{-\lambda}^{(k+l-a)}(\mu)\right.$$
$$\left.+\sum_{b=0}^{k}(-1)^l\binom{k}{b}f^{(b)}(-\lambda)\,\rho_\mu^{(k+l-b)}(-\lambda)\right\}$$
$$= -(-1)^{k+l}\sum_{a=0}^{l}\binom{l}{a}(k+l-a)!\,(-1)^a\,\frac{f^{(a)}(\mu)}{(\lambda+\mu)^{k+l+1-a}}$$
$$-(-1)^{k+l}\sum_{b=0}^{k}\binom{k}{b}(k+l-b)!\,(-1)^b\,\frac{f^{(b)}(\lambda)}{(\lambda+\mu)^{k+l+1-b}}. \tag{2.27}$$

Comparing this equation with (C.7), we see that when $\lambda+\mu\neq 0$

$$\frac{(-1)^{k+1}}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\, f(\xi)\,\rho_\xi^{(k)}(-\lambda)\,\rho_\xi^{(l)}(\mu) = -\left.\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\right|_{(\alpha,\beta)=(\lambda,\mu)}\frac{f(\alpha)+f(\beta)}{\alpha+\beta}. \tag{2.28}$$

Thus the proof of the lemma is complete. □

**Lemma 4.** *If* $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, *then*

$$(\nabla_X R)(\omega)Y = \sum_{k,l}\lim_{(\alpha,\beta)\to(\lambda,\mu)}\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\frac{f(\alpha+\beta)-f(\beta)}{\alpha}\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.29}$$

**Proof.** As a consequence of (A.3), the left-hand side of (2.29) can be written as

$$(\nabla_X R)(\omega)Y = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\ f(\xi)\,\rho_\xi(\bar\omega)\left[X, \rho_\xi(\bar\omega)\,Y\right]. \tag{2.30}$$

The application of the functional calculus (see also (2.14) and (C.6)) gives

$$\rho_\xi(\bar\omega)\left[X, \rho_\xi(\bar\omega)\,Y\right] = \sum_{k,l}\frac{\mathrm{d}^l}{\mathrm{d}\eta^l}\bigg|_{\eta=\mu}\rho_\xi^{(k)}(\lambda+\eta)\,\rho_\xi(\eta)\,\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.31}$$

Therefore,

$$(\nabla_X R)(\omega)\,Y = \sum_{k,l}\left\{\sum_{j=0}^{l}\binom{l}{j}\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\ f(\xi)\,\rho_\xi^{(k+l-j)}(\lambda+\mu)\,\rho_\xi^{(j)}(\mu)\right\}\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.32}$$

When $\lambda = 0$, the integrands have poles only at the point $\mu$. If $\lambda \neq 0$, then the integrands have poles at the points $\lambda+\mu$ and $\mu$.

*The $\lambda = 0$ case.* In this case $\rho_\xi^{(k+l-j)}(\lambda+\mu)\,\rho_\xi^{(j)}(\mu) = (k+l-j)!j!\,(\xi-\mu)^{-(k+l+1)-1}$. Thus

$$\begin{aligned}
&\sum_{j=0}^{l}\binom{l}{j}\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\ f(\xi)\,\rho_\xi^{(k+l-j)}(\lambda+\mu)\,\rho_\xi^{(j)}(\mu)\\
&\quad= \sum_{j=0}^{l}\binom{l}{j}(k+l-j)!j!\frac{f^{(k+l+1)}(\mu)}{(k+l+1)!} = \frac{k!l!f^{(k+l+1)}(\mu)}{(k+l+1)!}\sum_{j=0}^{l}\binom{(k+l)-j}{(k+l)-l}\\
&\quad= \frac{k!l!f^{(k+l+1)}(\mu)}{(k+l+1)!}\binom{k+l+1}{l} = \frac{f^{(k+l+1)}(\mu)}{k+1}\\
&\quad= \lim_{(\alpha,\beta)\to(\lambda,\mu)}\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\frac{f(\alpha+\beta)-f(\beta)}{\alpha}
\end{aligned} \tag{2.33}$$

where we used the combinatorial identity (B.2) and (C.14). So in this case (2.29) holds.

*The $\lambda \neq 0$ case.* Denote by $C_\alpha$ a sufficiently small circle around $\alpha \in \sigma(\bar\omega)$. Then, by Cauchy's theorem, the relevant integrals in (2.32) give

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}\xi\ f(\xi)\,\rho_\xi^{(k+l-j)}(\lambda+\mu)\,\rho_\xi^{(j)}(\mu) = \frac{1}{2\pi\mathrm{i}}\int_{C_\mu}\mathrm{d}\xi\ f(\xi)\,\rho_\xi^{(k+l-j)}(\lambda+\mu)\,\rho_\xi^{(j)}(\mu)\\
&\qquad+\frac{1}{2\pi\mathrm{i}}\int_{C_{\lambda+\mu}}\mathrm{d}\xi\ f(\xi)\,\rho_\xi^{(j)}(\mu)\,\rho_\xi^{(k+l-j)}(\lambda+\mu)\\
&\quad= \frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\bigg|_{\xi=\mu} f(\xi)\,\rho_\xi^{(k+l-j)}(\lambda+\mu) + \frac{\mathrm{d}^{k+l-j}}{\mathrm{d}\xi^{k+l-j}}\bigg|_{\xi=\lambda+\mu} f(\xi)\,\rho_\xi^{(j)}(\mu)\\
&\quad= (-1)^{k+l-j+1}\sum_{a=0}^{j}\binom{j}{a}(k+l-a)!\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}\\
&\qquad+\sum_{b=0}^{k+l-j}\binom{k+l-j}{b}(j+b)!\,(-1)^b\,\frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}}.
\end{aligned} \tag{2.34}$$

Thus the coefficient of $\left[b_\lambda^k X, b_\mu^l Y\right]/k!l!$ in (2.32) is equal to the following expression:

$$\sum_{j=0}^{l}\binom{l}{j}\left\{(-1)^{k+l-j+1}\sum_{a=0}^{j}\binom{j}{a}(k+l-a)!\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}\right.$$
$$\left.+\sum_{b=0}^{k+l-j}\binom{k+l-j}{b}(j+b)!\,(-1)^b\,\frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}}\right\}. \qquad (2.35)$$

Firstly, do the summation of the first part of (2.35):

$$\begin{aligned}
\text{Part }1(k,l) &:= \sum_{j=0}^{l}\binom{l}{j}(-1)^{k+l+1-j}\sum_{a=0}^{j}\binom{j}{a}(k+l-a)!\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}\\
&= (-1)^{k+l+1}\sum_{a=0}^{l}\frac{(k+l-a)!l!}{a!\,(l-a)!}\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}(-1)^a\sum_{j=0}^{l-a}\binom{l-a}{j}(-1)^j\\
&= (-1)^{k+l+1}\sum_{a=0}^{l}\frac{(k+l-a)!l!}{a!\,(l-a)!}\frac{f^{(a)}(\mu)}{\lambda^{k+l-a+1}}(-1)^a\,\delta_{l-a,0}\\
&= -(-1)^k\,k!\frac{f^{(l)}(\mu)}{\lambda^{k+1}}. \qquad (2.36)
\end{aligned}$$

Secondly, do the summation of the second part of (2.35). Introducing a new variable $m := j+b$, we obtain

$$\begin{aligned}
\text{Part }2(k,l) &:= \sum_{j=0}^{l}\binom{l}{j}\sum_{b=0}^{k+l-j}\binom{k+l-j}{b}(j+b)!\,(-1)^b\,\frac{f^{(k+l-j-b)}(\lambda+\mu)}{\lambda^{j+b+1}}\\
&= -\sum_{j=0}^{l}\sum_{b=0}^{k+l-j}(-1)^{m+1}\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}(-1)^j\binom{m}{j}\binom{k+l-j}{k}\\
&= -\sum_{m=0}^{l}(-1)^{m+1}\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{k+l-j}{k}\\
&\quad-\sum_{m=l+1}^{k+l}(-1)^{m+1}\frac{k!l!}{(k+l-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}\\
&\quad\times\sum_{j=0}^{l}(-1)^j\binom{m}{j}\binom{k+l-j}{k}. \qquad (2.37)
\end{aligned}$$

By means of the combinatorial identities (B.3), (B.10), we can simplify this formula. In fact, after a straightforward further computation, we get

$$\text{Part }2(k,l) = -\sum_{m=0}^{k}(-1)^{m+1}\frac{k!}{(k-m)!}\frac{f^{(k+l-m)}(\lambda+\mu)}{\lambda^{m+1}}. \qquad (2.38)$$

Now collecting equations (2.38), (2.36), (2.35), (2.32), in the $\lambda \neq 0$ case we can write

$$\begin{aligned}
(\nabla_X R)(\omega)\,Y &= \sum_{k,l}\{\text{Part }1(k,l)+\text{Part }2(k,l)\}\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}\\
&= \sum_{k,l}\frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l}\bigg|_{(\alpha,\beta)=(\lambda,\mu)}\frac{f(\alpha+\beta)-f(\beta)}{\alpha}\frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!} \qquad (2.39)
\end{aligned}$$

since equation (C.13) is valid. Hence lemma 4 is proved. □

**Lemma 5.** *If* $\lambda, \mu \in \sigma(\bar{\omega})$, $X \in N_\lambda$, $Y \in N_\mu$, *then*

$$(\nabla_Y R)(\omega)X = -\sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k \partial\beta^l} \frac{f(\alpha+\beta) - f(\alpha)}{\beta} \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}. \tag{2.40}$$

**Proof.** This is a trivial consequence of the preceding lemma. □

Now we are in the position to verify the mCDYBE (2.4) for the canonical $r$-matrix (2.7).

**Proof of theorem 1.** Let $\lambda, \mu \in \sigma(\bar{\omega})$ and $X \in N_\lambda$, $Y \in N_\mu$. By applying the four lemmas, the left-hand side of (2.4) can be written as

$$\begin{aligned}
&\tfrac{1}{4}[X, Y] + [R(\omega)X, R(\omega)Y] - R(\omega)([R(\omega)X, Y] + [X, R(\omega)Y]) \\
&\qquad + \langle X, (\nabla R)(\omega)Y\rangle + (\nabla_Y R)(\omega)X - (\nabla_X R)(\omega)Y \\
&\quad = \sum_{k,l} \lim_{(\alpha,\beta)\to(\lambda,\mu)} \frac{\partial^{k+l}}{\partial\alpha^k\partial\beta^l} \left( \frac{1}{4} + f(\alpha)f(\beta) - f(\alpha+\beta)(f(\alpha)+f(\beta)) \right. \\
&\qquad \left. - \frac{f(\alpha)+f(\beta)}{\alpha+\beta} - \frac{f(\alpha+\beta)-f(\alpha)}{\beta} - \frac{f(\alpha+\beta)-f(\beta)}{\alpha} \right) \frac{\left[b_\lambda^k X, b_\mu^l Y\right]}{k!l!}.
\end{aligned} \tag{2.41}$$

This equals zero since the 'addition formula' (C.1) is valid for the function $f$ in (1.3). □

## 3. Discussion

We have shown that the canonical $r$-matrix defined by (2.7) with $f$ in (1.3) satisfies the mCDYBE (2.4). It is worth noticing that our proof implies a uniqueness result as well. Suppose that we wish to define an antisymmetric solution of the mCDYBE (2.4) by the functional calculus, i.e., by using *some* holomorphic complex function in formula (2.7) now considered as an ansatz. For this formula to be well defined, the domain of holomorphicity of the function $f$ must contain zero, since this is always an eigenvalue of $\mathrm{ad}\,\omega$. Moreover, for $R$ to be antisymmetric, which is in turn necessary for the equivalence of (2.4) to (1.1) with $\varphi$ in (1.4), $f$ must be an odd function. Under these assumptions, the mCDYBE (2.4) for the ansatz (2.7) is in fact equivalent to the functional equation (C.1) for the unknown function $f$. Indeed, the whole calculation described in section 2 is valid for such an ansatz up to the equality in (2.41). The point then is that the functional equation (C.1) has a unique odd solution around the origin. The proof of this statement is quite easy. By taking the $y \to 0$ limit in (C.1) we obtain the differential equation for $f$ which appears in (C.2). With the initial value $f(0) = 0$, which is implied by $f$ being odd, this differential equation has a unique, holomorphic solution around the origin, namely the function $f(x) = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x}$.

So far we assumed the Lie algebra $\mathcal{G}$ to be complex, but the mCDYBE can be considered for a real self-dual Lie algebra, too. The real case arises naturally in applications [3,4]. Let us now suppose that $\mathcal{G}$ is the complexification of a real self-dual Lie algebra, say $\mathcal{G}_r$. Then it is not difficult to see that $R(\omega)$ given by (2.7) maps $\mathcal{G}_r$ to $\mathcal{G}_r$ if $\omega \in \mathcal{G}_r$. This is obviously the case if $\omega$ is near to zero, where one can apply the power series expansion of $f$ around zero to define $R(\omega)$. More generally, if $\omega \in \mathcal{G}_r$ then one may take the curve $C$ in (2.7) to be invariant under complex conjugation as the eigenvalues of ad $\omega$ occur in conjugate pairs. By using this and $f(\bar{z}) = \bar{f}(z)$, complex conjugation of (2.7) shows that $R(\omega)X \in \mathcal{G}_r$ if $\omega \in \mathcal{G}_r$ and $X \in \mathcal{G}_r$. Thus the canonical $r$-matrix is a solution of the mCDYBE (2.4) in the real case as well.

Our use of the functional calculus, which is applicable to Banach algebras in general [9], in the definition (2.7) might serve as a starting point for future work towards generalizations of this canonical $r$-matrix to certain infinite-dimensional self-dual Banach Lie algebras. However, this represents a nontrivial problem since the above-presented proof of theorem 1 relies heavily on the finite dimensionality of $\mathcal{G}$.

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## Appendix A. Functional calculus of linear operators

For convenient reference in the main text, in this appendix we collect some results from the theory of bounded operators based on chapter VII of the book [9].

Let $X \neq \{0\}$ be a complex Banach space. The space of bounded linear operators on $X$ is denoted by $B(X)$, which is a Banach algebra in the usual way. Let $T \in B(X)$ be a bounded linear operator. The resolvent set of $T$ is given by $\mathcal{R}(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ invertible operator}\}$, where $I$ is the unit operator. The spectrum $\sigma(T)$ of $T$ is the complement of $\mathcal{R}(T)$. The formula $\mathcal{R}(T) \ni \xi \mapsto \rho_\xi(T) = (\xi I - T)^{-1}$ defines the resolvent function of $T$. Denote by $\mathcal{F}(T)$ the set of all complex functions $f$ that are holomorphic on some neighbourhood of $\sigma(T)$. Then one can define the functions $f(T)$ of the operator $T$ as follows.

**Definition A.1.** *Let $f \in \mathcal{F}(T)$ and consider a closed, rectifiable curve $C$ that lies in the domain of analyticity of $f$ and encircles the spectrum $\sigma(T)$ in the positive sense customary in the theory of complex variables. Then the operator $f(T)$ is defined by the equation*

$$f(T) = \frac{1}{2\pi \mathrm{i}} \int_C f(\xi)\, \rho_\xi(T)\, \mathrm{d}\xi. \tag{A.1}$$

It can be shown that $f(T)$ depends only on the function $f$, and not on the curve $C$. Some important properties of this functional calculus are gathered in the following theorem.

**Theorem A.2.** *If $f, g \in \mathcal{F}(T)$ and $\alpha, \beta \in \mathbb{C}$ then*

- *$\alpha f + \beta g \in \mathcal{F}(T)$ and $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$,*
- *$fg \in \mathcal{F}(T)$ and $(fg)(T) = f(T)\, g(T)$,*
- *if $f$ has the power series expansion $f(z) = \sum_{k=0}^\infty c_k z^k$ valid in a neighbourhood of $\sigma(T)$, then $f(T) = \sum_{k=0}^\infty c_k T^k$.*

One can define the directional derivatives, $(\nabla_S f)(T) \in B(X)$, of $f(T)$ by

$$(\nabla_S f)(T) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(T + tS) \qquad S \in B(X). \tag{A.2}$$

The integral formula (A.1) implies the equation

$$(\nabla_S f)(T) = \frac{1}{2\pi \mathrm{i}} \int_C f(\xi) \rho_\xi(T) S \rho_\xi(T)\, \mathrm{d}\xi. \tag{A.3}$$

Now suppose that $X$ is a finite-dimensional Banach space. In this case the spectrum $\sigma(T)$ of the operator $T$ has finitely many elements, which are just the eigenvalues of $T$. The *index* $\nu(\lambda)$ of an eigenvalue $\lambda$ is the smallest positive integer $\nu$ such that $(\lambda I - T)^\nu x = 0$ for every vector $x$ for which $(\lambda I - T)^{\nu+1} x = 0$. Introducing the invariant subspaces $N_\lambda := \mathrm{Ker}(T - \lambda I)^{\nu(\lambda)}$, one has the usual $X = \oplus_{\lambda \in \sigma(T)} N_\lambda$ Jordan decomposition of $X$.

**Theorem A.3.** *If* $\dim(X) < \infty$ *and* $f \in \mathcal{F}(T)$*, then*

$$f\,(T) = \sum_{\lambda\in\sigma(T)} \sum_{k=0}^{\nu(\lambda)-1} \frac{1}{k!} f^{(k)}\,(\lambda)\,(T-\lambda I)^k\, E_\lambda \tag{A.4}$$

*where* $E_\lambda \in B(X)$ *is the projection operator of the subspace* $N_\lambda$.

## Appendix B. Some combinatorial identities

We here gather some elementary combinatorial identities needed in section 2.

**Identity B.1.** *If* $k, l \in \mathbb{N} := \{0, 1, 2, \ldots\}$*, then*

$$\sum_{n=0}^{k} (-1)^n \frac{1}{n+l+1} \binom{k}{n} = \frac{k!l!}{(k+l+1)!}. \tag{B.1}$$

**Proof.** By induction with respect to $k$. □

**Identity B.2.** *If* $k, n \in \mathbb{N}$ *and* $0 \leqslant k \leqslant n$*, then*

$$\sum_{a=0}^{k} \binom{n-a}{n-k} = \binom{n+1}{k}. \tag{B.2}$$

**Proof.** By induction with respect to $n$. □

**Identity B.3.** *Let* $k, l, m \in \mathbb{N}$ *and* $0 \leqslant m \leqslant l$*, then*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \binom{k+l-j}{k} = \begin{cases} 0 & \text{if } k < m \\ \binom{k+l-m}{l} & \text{if } k \geqslant m. \end{cases} \tag{B.3}$$

**Proof.** Consider the smooth function

$$\mathbb{R} \times (\mathbb{R} \setminus \{0\}) \ni (a, b) \mapsto b^{k+l-m}\,(a+b)^m\,. \tag{B.4}$$

Using the binomial theorem, we can write

$$b^{k+l-m}\,(a+b)^m = \sum_{j=0}^{m} \binom{m}{j} a^j b^{k+l-j}. \tag{B.5}$$

Let us differentiate this equation $k$ times with respect to $b$. Then the left-hand side gives

$$\begin{aligned} \frac{\partial^k}{\partial b^k}\left(b^{k+l-m}\,(a+b)^m\right) &= \sum_{i=0}^{k} \binom{k}{i} \left(\frac{\partial^{k-i}}{\partial b^{k-i}} b^{k+l-m}\right) \frac{\partial^i}{\partial b^i}\,(a+b)^m \\ &= \sum_{i=0}^{\min(m,k)} \binom{k}{i} \frac{(k+l-m)!\,m!}{(l+i-m)!\,(m-i)!} b^{l-m+i}\,(a+b)^{m-i}\,. \end{aligned} \tag{B.6}$$

By evaluating this equation at $a = -1$, $b = 1$, we obtain

$$\frac{\partial^k}{\partial b^k}\left(b^{k+l-m}\,(a+b)^m\right)\Big|_{a=-1,\,b=1} = \begin{cases} 0 & \text{if } k < m \\ k!\binom{k+l-m}{l} & \text{if } k \geqslant m. \end{cases} \tag{B.7}$$

At the same time, the right-hand side of (B.5) gives

$$\frac{\partial^k}{\partial b^k}\sum_{j=0}^{m}\binom{m}{j}a^j b^{k+l-j} = k!\sum_{j=0}^{m}\binom{m}{j}\binom{k+l-j}{k}a^j b^{l-j}. \tag{B.8}$$

It follows that

$$\frac{\partial^k}{\partial b^k}\sum_{j=0}^{m}\binom{m}{j}a^j b^{k+l-j}\bigg|_{a=-1,\,b=1} = k!\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{k+l-j}{k}. \tag{B.9}$$

Comparing (B.7) and (B.9) we see that our statement is valid. □

**Identity B.4.** *Let $k, l, m \in \mathbb{N}$ and $l < m \leqslant k + l$, then*

$$\sum_{j=0}^{l}(-1)^j\binom{m}{j}\binom{k+l-j}{k} = \begin{cases} 0 & \text{if } k < m \\ \binom{k+l-m}{l} & \text{if } k \geqslant m. \end{cases} \tag{B.10}$$

**Proof.** Similar to the preceding identity. □

## Appendix C. Addition formula and further identities

Let us consider the function $f(x) = \frac{1}{2}\coth\frac{x}{2} - \frac{1}{x}$. This function is holomorphic on the whole complex plane except the points $2\pi\mathrm{i}\mathbb{Z}^*$, where it has first-order poles. Using the familiar $\coth x \coth y - \coth(x+y)(\coth x + \coth y) + 1 = 0$ identity, the following 'addition formula' can be obtained:

**Identity C.1.** *If $x \neq 0$, $y \neq 0$, $x + y \neq 0$, then*

$$\frac{1}{4} + f(x)f(y) - f(x+y)(f(x)+f(y))$$
$$-\frac{f(x+y)-f(y)}{x} - \frac{f(x+y)-f(x)}{y} - \frac{f(x)+f(y)}{x+y} = 0. \tag{C.1}$$

On its domain of holomorphicity, the function $f$ satisfies also the relations

$$f^{(k)}(-x) = (-1)^{k+1}f^{(k)}(x) \qquad f'(x) + 2\frac{f(x)}{x} + f^2(x) = \frac{1}{4}. \tag{C.2}$$

The first relation in (C.2) uses only the fact that $f$ is an odd function, while the second relation follows, for example, by taking the $y \to 0$ limit in (C.1).

For convenience, we now collect some further identities that give the results for the differentiation of expressions of the type appearing in (C.1). All these identities are obvious, and are actually valid for any odd holomorphic function $f$. They are used in section 2 to derive the equality in (2.41) for the $r$-matrix of the form in (2.7).

**Identity C.2.** *If $k, l \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}\frac{1}{4} = \frac{1}{4}\delta_{k,0}\delta_{l,0} \tag{C.3}$$

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}f(x)f(y) = f^{(k)}(x)f^{(l)}(y) \tag{C.4}$$

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}f(x+y)f(x) = \frac{\mathrm{d}^k}{\mathrm{d}\xi^k}\bigg|_{\xi=x} f^{(l)}(\xi+y)f(\xi) \tag{C.5}$$

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l}f(x+y)f(y) = \frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\bigg|_{\xi=y} f^{(k)}(\xi+x)f(\xi). \tag{C.6}$$

**Identity C.3.** *If* $x + y \neq 0$, *then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^{k+l} \sum_{a=0}^{l} \binom{l}{a} (k + l - a)!\, (-1)^a \frac{f^{(a)}(y)}{(x + y)^{k+l+1-a}} + (-1)^{k+l} \sum_{b=0}^{k} \binom{k}{b} (k + l - b)!\, (-1)^b \frac{f^{(b)}(x)}{(x + y)^{k+l+1-b}}. \tag{C.7}$$

*We also have*

$$\lim_{x\to -y} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^k \frac{k!l!}{(k + l + 1)!} f^{(k+l+1)}(y). \tag{C.8}$$

**Proof.** Equation (C.7) is a direct consequence of the Leibniz rule. To verify (C.8), let us introduce $u := x + y$, $y = u - x$. By using power series expansion around $u = 0$, we have

$$\frac{f(x) + f(y)}{x + y} = \frac{f(x) + f(u - x)}{u} = \frac{f(x) - f(x - u)}{u} = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(x)}{(n + 1)!} u^n = \sum_{n=0}^{\infty} (-1)^n \frac{f^{(n+1)}(x)}{(n + 1)!} (x + y)^n. \tag{C.9}$$

Differentiating this equation $l$ times with respect to $y$, we get that

$$\frac{\partial^l}{\partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^l \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n + l + 1)} f^{(n+l+1)}(x)\,(x + y)^n. \tag{C.10}$$

Then differentiating $k$ times with respect to $x$, we obtain

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^l \sum_{n=0}^{k} \left( \frac{(-1)^n}{n + l + 1} \sum_{j=0}^{n} \binom{k}{j} \frac{f^{(n+l+1+k-j)}(x)}{(n - j)!} (x + y)^{n-j} \right) + (-1)^l \sum_{n=k+1}^{\infty} \left( \frac{(-1)^n}{n + l + 1} \sum_{j=0}^{k} \binom{k}{j} \frac{f^{(n+l+1+k-j)}(x)}{(n - j)!} (x + y)^{n-j} \right). \tag{C.11}$$

Now, let us take the limit $x \to -y$. Using the combinatorial identity (B.1), we can see that

$$\lim_{x\to -y} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x) + f(y)}{x + y} = (-1)^k f^{(k+l+1)}(y) \sum_{n=0}^{k} \frac{(-1)^n}{n + l + 1} \binom{k}{n} = (-1)^k \frac{k!l!}{(k + l + 1)!} f^{(k+l+1)}(y) \tag{C.12}$$

whereby the proof is complete. □

**Identity C.4.** *If* $x \neq 0$, *then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x + y) - f(y)}{x} = - \sum_{m=0}^{k} \frac{k!}{(k - m)!} (-1)^{m+1} \frac{f^{(k+l-m)}(x + y)}{x^{m+1}} - (-1)^k k! \frac{f^{(l)}(y)}{x^{k+1}}. \tag{C.13}$$

*In the limit case, we have*

$$\lim_{x\to 0} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x + y) - f(y)}{x} = \frac{f^{(k+l+1)}(y)}{k + 1}. \tag{C.14}$$

**Proof.** The verification of (C.13) is trivial. As for (C.14), the power series expansion of $f$ around $x = 0$ implies that

$$\frac{f(x+y) - f(y)}{x} = \frac{1}{1!} f'(y) + \cdots + \frac{1}{(k+1)!} f^{(k+1)}(y)\, x^k + \mathcal{O}\left(x^{k+1}\right). \tag{C.15}$$

By taking the derivatives of this equation, we obtain that

$$\frac{\partial^k}{\partial x^k} \frac{f(x+y) - f(y)}{x} = \frac{f^{(k+1)}(y)}{k+1} + \mathcal{O}(x) \tag{C.16}$$

and

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y) - f(y)}{x} = \frac{f^{(k+l+1)}(y)}{k+1} + \mathcal{O}(x) \tag{C.17}$$

which implies (C.14). □

**Identity C.5.** *If* $y \neq 0$*, then*

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y) - f(x)}{y} = -\sum_{m=0}^{l} \frac{l!}{(l-m)!} (-1)^{m+1} \frac{f^{(k+l-m)}(x+y)}{y^{m+1}} - (-1)^l\, l!\, \frac{f^{(k)}(x)}{y^{l+1}}. \tag{C.18}$$

*In the limit case,*

$$\lim_{y\to 0} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \frac{f(x+y) - f(x)}{y} = \frac{f^{(k+l+1)}(x)}{l+1}. \tag{C.19}$$

**Proof.** This is an obvious consequence of the preceding identity. □

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